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Shock Wave Boundary-Layer Interactions in Laminar Transonic Flow

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A solution is presented for a shock interacting with an unseparated laminar boundary layer in transonic flow. The method of matched asymptotic expansions is employed. The flowfield is divided into three regions: the transonic flow external to the boundary layer, an outer boundary-layer zone, and a thin viscous sublayer near the wall. The interaction occurs over a region with a length of order $Re^{-3/8}(M_\infty - 1)^{-3/8}$ and a height of order $Re^{-3/8}(M_\infty - 1)^{-7/8}$ where Re is the Reynolds number based on the boundary-layer length and M_∞ is the freestream Mach number. For the case of an incoming oblique shock, the shock is weakened as it approaches the boundary layer, an expansion fan is formed at the intersection of the shock and the boundary layer, and compression waves formed both upstream and downstream of the interaction coalesce to form the reflected shock. Numerical solutions for the wall pressure and other flow properties and a flow picture of the interaction region are presented for such a case. The results apply when $M_\infty - 1$ is between $O(Re^{-7/15})$ [limit when boundary-layer displacement effects dominate shock effects] and $O(Re^{-1/5})$ [boundary-layer separation limit].

Nomenclature

$a = \bar{a}/\bar{a}_\infty^*$ = nondimensional speed of sound
E = gage parameter, U-velocity component
$k = \bar{k}/\bar{k}_\infty^*$ = nondimensional thermal conductivity coefficient
$\bar{L}$ = distance, leading edge to shock intersection
$\bar{L}_x, \bar{L}_y$ = local characteristic lengths
M = Mach number
$P = \bar{P}/\bar{\rho}_\infty^* \bar{a}_\infty^{*2}$ = nondimensional pressure
$Pr^* = \bar{C}_p \bar{\mu}_\infty^*/\bar{k}_\infty^*$ = Prandtl number
$Re^* = \bar{\rho}_\infty^* \bar{a}_\infty^* \bar{L}/\bar{\mu}_\infty^*$ = Reynolds number
$T = \bar{T}/(\bar{a}_\infty^{*2}/\bar{C}_p)$ = nondimensional temperature
$U = \bar{U}/\bar{a}_\infty^*;\ V = \bar{V}/\bar{a}_\infty^*$ = nondimensional velocity components
$X = (\bar{X} - \bar{L})/\bar{L};\ Y = \bar{Y}/\bar{L}$ = nondimensional coordinates
$x; y$ = stretched coordinates
γ = ratio of specific heats
$\gamma(A, z) = \int_0^z s^{(A-1)} e^{-s} ds$ = incomplete gamma function
$\Gamma(A)$ = gamma function
ε = deviation of flow velocity from sonic
Λ' = shock strength parameter
$\mu = \bar{\mu}/\bar{\mu}_\infty^*$ = nondimensional viscosity coefficient
$\rho = \bar{\rho}/\bar{\rho}_\infty^*$ = nondimensional density
δ, ν, π, τ = gage parameters

Superscripts

$-$ = dimensional quantity
$*$ = sonic value
$\hat{}$ = inviscid rotational flow region
$+$ = viscous sublayer
$\sim$ = transonic flow region

Subscripts

∞ = incoming flow

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Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Shock Waves and Detonations; Subsonic and Transonic Flow.

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Introduction

IT is well known that a boundary layer greatly modifies the simple shock reflection at a solid surface and the attendant abrupt increase in surface pressure predicted by inviscid theory.[1,2] With a boundary layer included, the structure changes from a point of intersection of a shock and the surface, to a region of interaction between the shock and the boundary layer, and the pressure varies continuously.

In this paper, the interaction region between a shock wave and a boundary layer is studied for the case where the external flow is transonic and the boundary-layer flow is laminar and unseparated from a flat wall. The compelling reason for studying such an interaction problem is the application to transonic airfoil theory. Although the problem studied here does not meet all the requirements for such an application in that a turbulent boundary layer with a nonuniform outer flow over a curved body should be considered, the important mechanisms of the interaction will be illustrated. Much can be learned by consideration of the simpler problem of laminar flow over a flat plate studied here and by other authors.

Previous analyses of shock boundary-layer interactions have been concerned primarily with supersonic external flows. In the earliest theoretical studies,[3-5] and even more recent ones,[6] the flow was divided into two regions, the boundary layer and the external flow, with various approximations being made to simplify the boundary-layer calculations. A significant advance was made by Lighthill[7] when he divided the boundary layer into two regions consisting of an outer inviscid region, in which the flow is only slightly disturbed from the usual boundary-layer profile, and an inner viscous sublayer needed to insure the no-slip condition at the wall. Stewartson and Williams,[8] Feo,[9] and Messiter, Feo, and Melnik[10] employed this general model in their studies of flows separated by the interaction of an oblique shock and a laminar boundary layer, using the method of matched asymptotic expansions. Other approaches to the problem such as the application of integral methods[11,12] have resulted in very useful results which however are not locally exact and which therefore are not as useful in gaining an understanding of interaction phenomena.

The same general approach used in Refs. 8–10 is used here. Thus, the method of matched asymptotic expansions is employed and the interaction region is divided into three regions: the external inviscid transonic flow region, and the boundary-layer regions consisting of the outer inviscid rotational flow region and the viscous sublayer. Solutions in each region are presented in the form of asymptotic expansions valid in the limit as $Re^* \to \infty$ and $M_\infty - 1 \to 0$, where Re^* is the Reynolds number based on the distance to the point at which the shock intersects the boundary layer, and M_∞ is the Mach number of the external flow. These solutions are matched to insure the consistency of the solutions as the various regions are traversed. The results are applied to flow with an oblique shock wave such that the flow remains supersonic everywhere in the transonic flow regime. While the case where the shock is normal far from the boundary layer can also be handled with this model, it involves much more complex numerical computation[13] and will not be discussed in detail in this paper. Both cases are observed in flow over airfoils; the oblique shock wave serves as a good example of the method of solution.

Assumptions and Governing Equations

Consider a shock wave meeting a boundary layer at a distance $\bar{L}$ from the leading edge of a flat plate as shown in Fig. 1. The flow is assumed to be two dimensional and to consist of a perfect gas with constant specific heats, with viscosity coefficient proportional to the temperature, and with the Prandtl number equal to unity. The wall is taken to be insulated, i.e., the flow is adiabatic. As indicated in Fig. 1, the incoming uniform flow has a velocity of $\bar{U}_\infty = \bar{a}_\infty{}^*(1+\varepsilon)$ where $\varepsilon \ll 1$ because the flow is transonic.

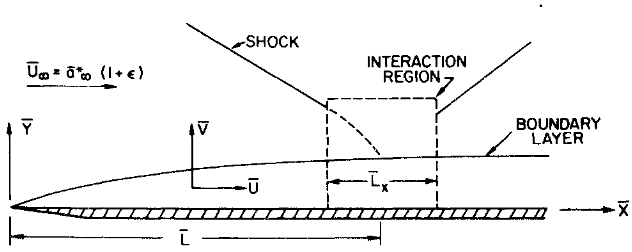

Fig. 1 Sketch showing flat plate, boundary layer, shock wave, interaction region, and notation used.

As a result of the preceding assumptions, the stagnation enthalpy is constant everywhere, and the boundary layer without an interaction would satisfy the compressible flow equivalent of the Blasius solution.[14] Within the interaction region, the flow quantities are perturbed from their values when no shock is present. For example, the U-velocity component in the boundary layer is a small perturbation from the compressible flow equivalent of the Blasius profile.

The governing equations are those of continuity, motion, energy, and state. They are written in nondimensional form as follows, where X is referred to $\bar{L}$ and the summation convention is used:

$$\partial \rho U_j / \partial X_j = 0 \tag{1a}$$

$$\rho U_j \frac{\partial U_i}{\partial X_j} + \frac{\partial P}{\partial X_i} = \frac{1}{Re^*}\frac{\partial}{\partial X_j}\left[\mu\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) + \left(\mu' - \frac{2}{3}\mu\right)\frac{\partial U_k}{\partial X_k}\delta_{ij}\right] \tag{1b}$$

$$T + (U_j{}^2/2) = (\gamma+1)/2(\gamma-1) \tag{1c}$$

$$P = [(\gamma-1)/\gamma]\rho T \tag{1d}$$

An equation which may be derived from Eqs. (1), the so-called gas dynamic equation, is very useful in transonic flows. It may be written as

$$\rho\left[U_j \frac{\partial}{\partial X_j}\left(\frac{U_i{}^2}{2}\right) - a^2 \frac{\partial U_j}{\partial X_j}\right] = -\frac{(\gamma-1)}{Re^*}\frac{\partial}{\partial X_j}\left(k\frac{\partial T}{\partial X_j}\right) + \frac{1}{Re^*}\left\{U_i \frac{\partial}{\partial X_j}\left[\mu\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)\right] + U_j \frac{\partial}{\partial X_j}\left[\left(\mu' - \frac{2}{3}\mu\right)\frac{\partial U_i}{\partial X_i}\right] - (\gamma-1)\left[\mu\frac{\partial U_i}{\partial X_j}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) + \left(\mu' - \frac{2}{3}\mu\right)\left(\frac{\partial U_j}{\partial X_j}\right)^2\right]\right\} \tag{2}$$

The boundary conditions are similar to those used in familiar boundary-layer problems. At the wall the velocity is zero; upstream of the interaction region, the solutions must match with those of the undisturbed boundary layer. Far from the wall, at least in the limit as $Re^* \to \infty$, solutions should match with the inviscid solution for the given shock. Downstream of the interaction and outside of the boundary layer, the disturbance dies out with the variables going to uniform post shock values. Finally, since the boundary-layer equations are parabolic, the solution in the downstream limit of the interaction region becomes the initial condition for the boundary-layer flow downstream of the interaction region.

In order to derive the specific governing equations valid in each region, it is necessary to stretch the independent variables such that each is of order unity in the region under consideration. The characteristic lengths in the X and Y directions, $\bar{L}_x$ and $\bar{L}_y$, respectively, are chosen to do this. Thus, $\bar{L}_y$ is considered to be of the order of the thickness of the region and changes, depending on the region being considered; however, it can easily be shown as a result of the matching conditions[13] that $\bar{L}_x$ is the same for all regions. The stretched variables are defined as follows

$$x = (\bar{X} - \bar{L})/\bar{L}_x = X/\tau \qquad \tau = \bar{L}_x/\bar{L} \tag{3a}$$

$$y = \bar{Y}/\bar{L}_y = (Y/\delta) \qquad \delta = \bar{L}_y/\bar{L} \tag{3b}$$

Superscripts are used to denote the region under consideration. Thus, $\tilde{y}$, $\hat{y}$, y^+ and $\tilde{\delta}$, $\hat{\delta}$, δ^+ are the stretched Y coordinate and gage parameter in the external, outer boundary-layer, and inner boundary-layer (sublayer) regions, respectively. Since τ and x do not change, no superscripts are necessary.

The dependent variables are expanded in asymptotic expansions valid in the limit as $Re^* \to \infty$ and $\varepsilon \to 0$. Constraints on the relative orders of Re^* and ε are shown later in terms of limitations on the solution. In the boundary-layer regions, the expansions are written as perturbations from boundary-layer solutions which themselves may be written as Taylor expansions.[13] Thus, in the outer boundary-layer region, the undisturbed boundary-layer solutions may be written as a Taylor expansion about $x = 0$, i.e., for $x \ll 1$, while in the inner boundary-layer region these solutions are expansions about $x = y = 0$, i.e., for $x \ll 1$ and $y \ll 1$. Therefore, the general solutions in the interaction region are written as double expansions, one for the undisturbed solution and one for the interaction effects. They are written as follows for the velocity components and pressure:

$$U = E_{oo}u_{oo}(x,y)+E_{o1}u_{o1}(x,y)+\cdots+E_1u_1(x,y)+E_2u_2(x,y)+\cdots \quad (4a)$$

$$V = v_{oo}v_{oo}(x,y)+v_{o1}v_{o1}(x,y)+\cdots+v_1v_1(x,y)+v_2v_2(x,y)+\cdots \quad (4b)$$

$$P = 1/\gamma+\cdots+\pi_1P_1(x,y)+\pi_2P_2(x,y)+\cdots \quad (4c)$$

where double subscripts refer to the known boundary-layer solution and single subscripts to the unknown interaction effects. Similar expansions may be written for the temperature and density. The expansions for the external transonic flow region are similar to those shown in Eqs. (4) but generally require only a single expansion involving only interaction terms. The gage parameters E, v, and π will be found in terms of Re^* and ε; with the exception of E_{oo} they are small compared to unity. As with γ and δ, superscripts are used to distinguish the dependent variables and gage parameters in each region.

Solutions

Inviscid Rotational Flow Region

The flow in this outer boundary-layer region is a small perturbation from the boundary-layer flow at the point at which the shock meets the boundary layer. Expansions (4) simplify to

$$U = \hat{u}_{oo}(\hat{y})+\tau x\hat{u}_{o1}(\hat{y})+\cdots+\hat{E}_1\hat{u}_1(x,\hat{y})+\hat{E}_2\hat{u}_2(x,\hat{y})+\cdots \quad (5a)$$

$$V = \hat{\delta}\hat{v}_{oo}(\hat{y})+\hat{\delta}\tau x\hat{v}_{o1}(\hat{y})+\cdots+\hat{v}_1\hat{v}_1(x,\hat{y})+\hat{v}_2\hat{v}_2(x,\hat{y})+\cdots \quad (5b)$$

$$P = (1/\gamma)+\cdots+\hat{\pi}_1\hat{P}_1(x,\hat{y})+\hat{\pi}_2\hat{P}_2(x,\hat{y})+\cdots \quad (5c)$$

with expansions similar to Eq. (5a) for the temperature and density.[13] (That the gage parameters are the same for T and ρ is obvious upon consideration of the continuity, state, and energy equations.) Since this region comprises most of the boundary layer region

$$\delta = Re^{*-1/2} \quad (6)$$

The gage parameters are related by substituting Eqs. (5) into Eqs. (1). First, in the continuity equation, variations in both the x and y directions appear to be equally important. Next, comparing the viscous stress and pressure terms in the x-momentum equation, we find that the viscous terms can be ignored, i.e., that to the order considered, the perturbations are governed by inviscid equations where inertia and pressure terms balance each other. Thus, from the continuity and x-momentum equations we obtain, respectively,

$$\hat{E}_i/\tau = \hat{v}_i/\hat{\delta} \quad (7a)$$

$$\hat{\pi}_i = \hat{E}_i \quad (7b)$$

where the equality is taken for convenience and $i = 1, 2$. That is, two terms of the expansion are required for matching with the other regions. The y-momentum equation simplifies to

$$\partial\hat{P}_i/\partial\hat{y} = O[(\hat{\delta}/\tau)^2]$$

and since, as will be shown later, $\hat{\delta} \ll \tau$, this equation simply reduces to $\hat{P}_i = \hat{P}_i(x)$. In view of these arguments, Eqs. (1) become

$$\hat{\rho}_{oo}\,\partial\hat{u}_i/\partial x+\hat{u}_{oo}\,\partial\hat{\rho}_i/\partial x+(\partial/\partial\hat{y})(\hat{\rho}_{oo}\hat{v}_i) = 0 \quad (8a)$$

$$\hat{\rho}_{oo}\hat{u}_{oo}\,\partial\hat{u}_i/\partial x+\hat{\rho}_{oo}\hat{v}_i\,d\hat{u}_{oo}/d\hat{y}+d\hat{P}_i/dx = 0 \quad (8b)$$

$$\hat{T}_i+\hat{u}_{oo}\hat{u}_i = 0 \quad (8c)$$

$$\hat{P}_i = [(\gamma-1)/\gamma][\hat{\rho}_{oo}\hat{T}_i+\hat{T}_{oo}\hat{\rho}_i]+(1-i) \quad (8d)$$

After some manipulation, the solutions for $\hat{u}_i$ and $\hat{v}_i$ can be shown to be

$$\hat{u}_i = \hat{A}_i(x)\frac{d\hat{u}_{oo}}{d\hat{y}}+\left[\frac{d\hat{u}_{oo}}{d\hat{y}}\int_{\hat{y}}^{\infty}\left(\frac{1-\hat{M}_{oo}^2}{\hat{M}_{oo}^2}\right)d\hat{y}-\frac{1}{\hat{\rho}_{oo}\hat{u}_{oo}}\right]\cdot[\hat{P}_i(x)-\hat{P}_i(-\infty)] \quad (9a)$$

$$\hat{v}_i = -\frac{d\hat{A}_i}{dx}\hat{u}_{oo}-\hat{u}_{oo}\int_{\hat{y}}^{\infty}\left(\frac{1-\hat{M}_{oo}^2}{\hat{M}_{oo}^2}\right)d\hat{y}\cdot\frac{d\hat{P}_i}{dx} \quad (9b)$$

Here $\hat{M}_{oo} = \hat{u}_{oo}/\hat{a}_{oo}M_\infty$ where $M_\infty = 1+\varepsilon(\gamma+1)/2$ so that as $\hat{y}\to\infty$, $\hat{M}_{oo}\to 1$. $\hat{A}_i(x)$ is an arbitrary function of integration. These solutions are similar to those found in Ref. 8.

For matching purposes, it is necessary to find the limits of the preceding solutions as $\hat{y}\to 0$ and as $\hat{y}\to\infty$. It can be shown[13] that as $\hat{y}\to 0$

$$\int_{\hat{y}}^{\infty}\left(\frac{1-\hat{M}_{oo}^2}{\hat{M}_{oo}^2}\right)d\hat{y} = \frac{(\gamma+1)}{2a_1'^2}\frac{1}{\hat{y}}+O(1)$$

$$a_1' = \left[\frac{d\hat{u}_{oo}}{d\hat{y}}\right]_{\hat{y}=0} = \frac{2}{(\gamma+1)}a_1 = \frac{2}{(\gamma+1)}(0.33206)$$

Hence, as $\hat{y}\to 0$, the solutions become

$$U = a_1'\hat{y}+\frac{(\gamma-1)}{3(\gamma+1)}a_1'^3\hat{y}^3+\cdots+\tau x\left[-\frac{a_1'}{2}\hat{y}+\cdots\right]+\hat{E}_1\{\hat{A}_1(x)[a_1'+\cdots]+[\hat{P}_1(x)-\hat{P}_1(-\infty)][O(1)]\}+\hat{E}_2\{\hat{A}_2(x)[a_1'+\cdots]+[\hat{P}_2(x)-\hat{P}_2(-\infty)][O(1)]\}+\cdots \quad (10a)$$

$$V = \hat{\delta}\left[\frac{a_1'}{4}\hat{y}^2+\cdots\right]+\frac{\hat{\delta}}{\tau}\hat{E}_1\left\{-\frac{d\hat{A}_1}{dx}a_1'\hat{y}+\cdots-\frac{(\gamma+1)}{2a_1'}\frac{d\hat{P}_1}{dx}+\cdots\right\}+\frac{\hat{\delta}}{\tau}\hat{E}_2\left\{-\frac{d\hat{A}_2}{dx}a_1'\hat{y}+\cdots-\frac{(\gamma+1)}{2a_1'}\frac{d\hat{P}_2}{dx}+\cdots\right\} \quad (10b)$$

$$P = (1/\gamma)+\cdots+\hat{E}_1\hat{P}_1(x)+\hat{E}_2\hat{P}_2(x)+\cdots \quad (10c)$$

As $\hat{y}\to\infty$, the solutions become

$$U = 1+\varepsilon+\cdots-\hat{E}_1[\hat{P}_1(x)-\hat{P}_1(-\infty)]-\hat{E}_2[\hat{P}_2(x)-\hat{P}_2(-\infty)]+\cdots \quad (11a)$$

$$V = \hat{\delta}\hat{v}_{oo}(\infty)+\hat{\delta}\tau x\hat{v}_{o1}(\infty)+\cdots-(\hat{\delta}/\tau)\hat{E}_1\,d\hat{A}_1/dx-(\hat{\delta}/\tau)\hat{E}_2\,d\hat{A}_2/dx+\cdots \quad (11b)$$

$$P = (1/\gamma)+\cdots+\hat{E}_1\hat{P}_1(x)+\hat{E}_2\hat{P}_2(x)+\cdots \quad (11c)$$

In Eqs. (10) and (11), the expansions for $\hat{u}_{oo}$ are the Taylor expansions of the compressible boundary-layer solutions[13,14] mentioned previously.

Viscous Sublayer Region

The solution in the inviscid rotation flow does not satisfy the no-slip boundary condition, as shown by Eq. (10a). Very near the wall, then, there exists a thinner region ($\delta^+ \ll \hat{\delta}$) in which viscous stresses are as important as the inertia and pressure terms. Using expansions for the compressible boundary-layer flow solutions,[13,14] one can write Eqs. (4) in inner variables as follows:

$$U = \frac{\delta^+}{\hat{\delta}}a_1'y^+ +\left(\frac{\delta^+}{\hat{\delta}}\right)^3\frac{(\gamma-1)}{3(\gamma+1)}a_1'^3y^{+3}-\tau\frac{\delta^+}{\hat{\delta}}\frac{a_1'}{2}xy^+ +\cdots+E_1^+u_1^+(x,y^+)+\cdots \quad (12a)$$

$$V = \hat{\delta}(\delta^+/\hat{\delta})^2(a_1'/4)y^{+2}+\cdots+v_1^+v_1^+(x,y^+)+\cdots \quad (12b)$$

$$P = (1/\gamma)+\cdots+\pi_1^+P_1^+(x,y^+)+\cdots \quad (12c)$$

where only one interaction term is necessary in this inner boundary-layer region. Since $U \ll 1$ and the stagnation enthalpy is constant, the temperature and hence the density are constant to first order, i.e., the flow in the viscous sublayer is incompressible to first order.

Just as in the previous calculations, Eqs. (12) are substituted into Eqs. (1). Again, it is expected that variations in both x and y^+ are important insofar as the continuity equation is concerned. In the x-momentum equation, the inertia, viscous, and pressure terms are of the same order. Hence, from the continuity equation

$$E_1^+/\tau = v_1^+/\delta^+ \quad (13)$$

while from the x momentum equation

$$\pi_1{}^+ = (\delta^+/\hat{\delta})E_1{}^+ \quad (14a)$$

$$\delta^+ = (\hat{\delta}\tau/Re^*)^{1/3} \quad (14b)$$

Again, from the y^+ momentum equation, since $\delta^+ \ll \hat{\delta} \ll \tau$, $P_1{}^+ = P_1{}^+(x)$. Finally, then, the governing equations are

$$(\partial u_1{}^+/\partial x)+(\partial v_1{}^+/\partial y^+) = 0 \quad (15a)$$

$$\frac{2}{(\gamma+1)}a_1'\left[y^+\frac{\partial u_1{}^+}{\partial x}+v_1{}^+\right]+\frac{dP_1{}^+}{dx} = \left(\frac{\gamma+1}{2}\right)\frac{\partial^2 u_1{}^+}{\partial y^{+2}} \quad (15b)$$

Thus, Eqs. (15) are linearized boundary-layer equations. Differentiating Eq. (15b) with respect to y^+ and employing Eq. (15a), we obtain

$$\Lambda y^+(\partial^2 u_1{}^+/\partial x\,\partial y^+) = \partial^3 u_1{}^+/\partial y^{+3} \quad (16a)$$

$$\Lambda = 4a_1'/(\gamma+1)^2 \quad (16b)$$

If $\partial u_1{}^+/\partial y^+$ is replaced by z, say, Eq. (16a) is a specific case of a general equation integrated by Sutton.[15] Using Sutton's results, we can derive the following results

$$u_1{}^+ = -c_1\int_{-\infty}^{x}\frac{dP_1{}^+(\zeta)}{d\zeta}(x-\zeta)^{-1/3}\gamma\left(\frac{1}{3},\frac{\Lambda y^{+3}}{9(x-\zeta)}\right)d\zeta \quad (17a)$$

$$v_1{}^+ = c_1 y^+\int_{-\infty}^{x}(x-\zeta)^{-1/3}\frac{d^2P_1{}^+(\zeta)}{d\zeta^2}\gamma\left(\frac{1}{3},\frac{\Lambda y^{+3}}{9(x-\zeta)}\right)d\zeta - \left(\frac{9}{\Lambda}\right)^{1/3}c_1\int_{-\infty}^{x}\frac{d^2P_1{}^+(\zeta)}{d\zeta^2}\gamma\left(\frac{2}{3},\frac{\Lambda y^{+3}}{9(x-\zeta)}\right)d\zeta \quad (17b)$$

where $\gamma(A,z)$ is the incomplete Gamma function. Clearly, the boundary conditions $u_1{}^+ = v_1{}^+ = 0$ at $y^+ = 0$ are met by these solutions. For matching purposes, it is necessary to find their limiting forms as $y^+ \to \infty$. When these limiting forms and Eqs. (13) and (14) are subsituted into Eqs. (12), the results are, for $y^+ \to \infty$

$$U = (\delta^+/\hat{\delta})a_1'y^+ + (\delta^+/\hat{\delta})^3[(\gamma-1)/3(\gamma+1)]a_1'^3y^{+3} - \tau^+(\delta^+/\hat{\delta})(a_1'/2)xy^+ + E_1{}^+\left[-c_1\Gamma(1/3)\int_{-\infty}^{x}\frac{dP_1{}^+(\zeta)}{d\zeta}(x-\zeta)^{-1/3}d\zeta+\cdots\right]+\cdots \quad (18a)$$

$$V = \hat{\delta}(\delta^+/\hat{\delta})^2(a_1'/4)y^{+2}+\cdots+(\delta^+/\tau)E_1{}^+\times\left[c_1y^+\Gamma(1/3)\int_{-\infty}^{x}(x-\zeta)^{-1}\frac{d^2P_1{}^+}{d\zeta^2}d\zeta - \left(\frac{9}{\Lambda}\right)^{1/3}\Gamma(2/3)c_1\frac{dP_1{}^+(x)}{dx}+\cdots\right]+\cdots \quad (18b)$$

$$P = (1/\gamma)+\cdots+(\delta^+/\hat{\delta})E_1{}^+P_1{}^+(x)+\cdots \quad (18c)$$

where $dP_1{}^+/dx \to 0$ as $x \to -\infty$.

Transonic Flow Region

The flow outside the boundary layer is transonic. Hence, the flow quantities are small perturbations from their critical, undisturbed, values. Thus

$$U = 1+\tilde{E}_1\tilde{u}_1(x,\tilde{y})+\cdots \quad (19a)$$

$$V = \tilde{v}_1\tilde{v}_1(x,\tilde{y})+\cdots \quad (19b)$$

$$P = (1/\gamma)+\tilde{\pi}_1\tilde{P}_1(x,\tilde{y})+\cdots \quad (19c)$$

with similar expansions for ρ and T from their critical values of 1 and $(\gamma-1)^{-1}$, respectively. If these expansions and the stretched independent (outer) variables are substituted into Eqs. (1) and (2), the gas dynamic equation, it can be shown[13] that the viscous terms are negligible to first order and that in fact in first order, the familiar equations for inviscid irrotational transonic flow are obtained. The transonic equation, irrotationality condition, and pressure velocity relation are

$$[(\gamma+1)/2](\partial^2\tilde{u}_1{}^2/\partial x^2)-(\partial^2\tilde{u}_1/\partial\tilde{y}^2) = 0 \quad (20a)$$

$$(\partial\tilde{u}_1/\partial\tilde{y}) = (\partial\tilde{v}_1/\partial x) \quad (20b)$$

$$\tilde{P}_1 = -\tilde{u}_1 \quad (20c)$$

In addition, the relationships between the gage factors are

$$\tilde{v}_1 = \tilde{E}_1{}^{3/2} \quad (21a)$$

$$\tilde{\pi}_1 = \tilde{E}_1 \quad (21b)$$

$$\tau = \hat{\delta}\tilde{E}_1{}^{1/2} \quad (21c)$$

all well known results in transonic flow.

The solution to Eqs. (20) depends on the problem being studied (e.g., oblique or normal shock) as will be illustrated later.

If Eqs. (19) are matched with the corresponding properties of the uniform incoming outer flow, it is seen that as $x \to -\infty$

$$\tilde{u}_1(-\infty,\tilde{y}) = \tilde{P}_1(-\infty,\tilde{y}) = 1 \quad (22a)$$

$$\tilde{E}_1 = \varepsilon \quad (22b)$$

$$\tilde{v}_1 = \varepsilon^{3/2} \quad (22c)$$

Then, since $\tilde{u}_1$, $\tilde{P}_1$, etc., remain of order one in the entire region, the solutions to be used for matching with the inner region are, as $y \to 0$

$$U = 1+\varepsilon\tilde{u}_1(x,0)+\cdots \quad (23a)$$

$$V = \varepsilon^{3/2}\tilde{v}_1(x,0)+\cdots \quad (23b)$$

$$P = (1/\gamma)+\varepsilon\tilde{P}_1(x,0)+\cdots \quad (23c)$$

Matching

In order to complete the general solution, it is necessary to match solutions valid in each of the individual regions. Thus, Eqs. (10), valid in the limit as $\hat{y} \to 0$ must match term by term with Eqs. (18), valid in the limit as $y^+ \to \infty$. Likewise, Eqs. (11), valid in the limit as $\hat{y} \to \infty$, must match with Eqs. (23), valid in the limit as $\tilde{y} \to 0$.

Comparing first, Eqs. (10a) and (18a), we see that

$$E_1{}^+ = \hat{E}_1 \quad (24)$$

Then if the pressures, Eqs. (10c) and (18c), are to match

$$\hat{P}_1(x) = 0 \quad (25a)$$

$$\hat{E}_2 = (\delta^+/\hat{\delta})E_1{}^+ \quad (25b)$$

$$\hat{P}_2(x) = P_1{}^+(x) \quad (25c)$$

Returning to the velocity components, Eqs. (10a, b) and (18a, b), we see that matching is completed if

$$a_1'\hat{A}_1(x) = -c_1\Gamma(1/3)\int_{-\infty}^{x}\frac{dP_1{}^+(\zeta)}{d\zeta}(x-\zeta)^{-1/3}d\zeta \quad (26a)$$

$$(9/\Lambda)^{1/3}\Gamma(2/3)c_1 = (\gamma+1)/2a_1' \quad (26b)$$

Next, comparing Eqs. (11c) and (23c), we obtain

$$\hat{E}_2 = \varepsilon \quad (27a)$$

$$\tilde{P}_1(x,0) = \hat{P}_2(x) \quad (27b)$$

In view of Eq. (20c), and since $\hat{P}_2(-\infty) = -1$, it is seen that the U-velocity components, Eqs. (11a) and (23a), also match. Finally, from matching Eqs. (11b) and (23b), we obtain

$$(\hat{\delta}/\tau)\hat{E}_1 = \varepsilon^{3/2} \quad (28a)$$

$$\tilde{v}_1(x,0) = -d\hat{A}_1/dx \quad (28b)$$

Thus, it is seen that $\hat{A}_2(x)$ is not important in determining the first-order solution in each region.

It should be noted that Eq. (19b) is written with the idea that the undisturbed boundary-layer displacement effects are small compared to interaction effects and can be neglected. Now, if the boundary-layer effects are included, the result is,

$$V = \hat{\delta}\hat{v}_{oo}(\infty)+\hat{\delta}\tau x\hat{v}_{o1}(\infty)+\cdots+\varepsilon^{3/2}\tilde{v}_1(x,0)+\cdots \quad (29)$$

A constant in this expansion for V presents no problems. (The outer edge of the boundary layer acts like an inclined flat plate.) Hence, the present solution holds as long as the second boundary-layer displacement term is small compared to the first interaction term, i.e., as long as $\hat{\delta}\tau \ll \varepsilon^{3/2}$. When these terms are of the same order, modifications must be made; however, for this case the shock is extremely weak, and it will not be discussed further.

From Eqs. (14b, 24, 25b, 27a, and 28a) it is seen that

Table 1 Summary of gage parameters

Parameter	Viscous sublayer region (+′)	Inviscid rotational flow region (ˆ)	Transonic flow region (~)
δ	$Re^{*-5/8}\varepsilon^{-1/8}$	$Re^{*-1/2}$	$Re^{*-3/8}\varepsilon^{-7/8}$
E_{oo}	$Re^{*-1/8}\varepsilon^{-1/8}$	1	1
E_1	$Re^{*1/8}\varepsilon^{9/8}$	$Re^{*1/8}\varepsilon^{9/8}$	ε
E_2	—	ε	—
v_{oo}	$Re^{*-3/4}\varepsilon^{-1/4}$	$Re^{*-1/2}$	$Re^{*-1/2}$
v_1	$Re^{*-1/8}\varepsilon^{1/8}$	$\varepsilon^{3/2}$	$\varepsilon^{3/2}$
π_{oo}	1	1	1
π_1	ε	ε	ε
τ	←	$Re^{*-3/8}\varepsilon^{-3/8}$	→

$$\tau = Re^{*1/4}\hat{\delta}^{5/4}\varepsilon^{-3/8} \tag{30}$$

This is a general expression for the size of the interaction region in terms of the boundary-layer thickness at the point of intersection of the shock. Employing Eq. (6), we obtain

$$\tau = Re^{*-3/8}\varepsilon^{-3/8} \tag{31}$$

A summary of the remaining gage parameters in presented in Table 1. It should be noted that τ is large compared both to $\hat{\delta}$ and the shock thickness which is of order $Re^{*-1}\varepsilon^{-1}$. The solutions are valid within certain limits on the parameters. The lower limit, discussed previously, occurs when $\delta\tau = O(\varepsilon^{3/2})$, i.e., when $\varepsilon = O(Re^{*-7/15})$. The upper limit occurs when the viscous sublayer is no longer linear $[\delta^+/\hat{\delta} = O(E_1{}^+)$ from Eq. (12a)]. This yields $\varepsilon = O(Re^{*-1/5})$. Since $u_1{}^+$ is negative, this limit is that for which the shock wave is strong enough to cause separation. This result agrees with the conclusions of Messiter, Feo, and Melnik.[10]

Application to Oblique Shock Problem

An oblique shock impinging upon the boundary layer is considered. Because the shock has too great a pressure gradient for the boundary layer to sustain in the thickness of the shock, the pressure change is spread out over a distance greater than either the shock or the boundary-layer thickness, as shown previously. Thus, the pressure begins to increase ahead of the shock boundary-layer intersection point and continues to increase downstream of it. Compression characteristics ahead of the shock interact with it; the shock is weakened, but still has a finite strength at the impingement point. Hence, at that point a centered expansion fan is formed, to insure continuity of pressure.

The case considered is that where the flow remains supersonic everywhere in the transonic flow region; hence the method of characteristics can be applied. If $\xi(x,\tilde{y})$ and $\eta(x,\tilde{y})$ are coordinates which are constant on c^- and c^+ characteristics, respectively, then the equations for the c^- and c^+ characteristics, are, respectively,

$$(d\tilde{y}/dx)_{\xi=\text{const}} = -[(\gamma+1)\tilde{u}_1]^{-1/2} \tag{32a}$$

$$(d\tilde{y}/dx)_{\eta=\text{const}} = [(\gamma+1)\tilde{u}_1]^{-1/2} \tag{32b}$$

The equations which hold along the c^+ and c^- characteristics, respectively, are[13]

$$\tfrac{2}{3}(\gamma+1)^{1/2}\tilde{u}_1{}^{3/2} - \tilde{v}_1 = f_1(\eta) \tag{33a}$$

$$\tfrac{2}{3}(\gamma+1)^{1/2}\tilde{u}_1{}^{3/2} + \tilde{v}_1 = f_2(\xi) \tag{33b}$$

Now, the c^- characteristics ahead of the shock come from a region of uniform flow with $\tilde{u}_1 = 1$ and $\tilde{v}_1 = 0$. Therefore, ahead of the shock is a region of simple wave flow in the sense used in supersonic flow theory. Thus, from Eq. (33b) and the boundary conditions, we find that

$$f_2(\xi) = \tfrac{2}{3}(\gamma+1)^{1/2} \tag{34}$$

ahead of the wave, and hence from Eqs. (33) and (32b) we find that c^+ characteristics are straight lines with the parametric equation

$$[(\gamma+1)\tilde{u}_1(x_o,0)]^{1/2}\tilde{y} = x - x_o \tag{35}$$

Here, x_o is the value of x at which the characteristic leaves the boundary layer.

Applying matching conditions given by Eqs. (27b, 28b, 26a, and 20c) to Eqs. (33b) and (34), we obtain an equation for the pressure along the wall, ahead of the impingement point

$$[-\hat{P}_2(x)]^{3/2} + \omega\int_{-\infty}^{x}(x-\zeta)^{-1/3}\frac{d^2\hat{P}_2(\zeta)}{d\zeta^2}d\zeta = 1 \quad (x<0) \tag{36a}$$

$$\omega = 3^{1/3}\Gamma(\tfrac{1}{3})/2^{4/3}a_1{}'^{5/3}(\gamma+1)^{1/6}\Gamma(\tfrac{2}{3}) \tag{36b}$$

Equation (36a) is solved using a series of exponentials (similar to a series used by Stewartson and Williams[8])

$$\hat{P}_2(x) = -1 + \sum_1^{\infty}\tilde{a}_n e^{nbx} \tag{37}$$

where

$$b = \{\omega\Gamma(\tfrac{5}{3})\}^{-3/4} \tag{38a}$$

$$\tilde{a}_2 = -\tilde{a}_1{}^2/4(2^{4/3}-1) = \tilde{A}_2\tilde{a}_1{}^2 \tag{38b}$$

$$\tilde{a}_n = \left[\frac{\tfrac{3}{4}(n-1)^{4/3}-1}{n^{4/3}-1}\right]\tilde{a}_1\tilde{a}_{n-1} + \sum_{j=2}^{n=1}\tilde{a}_{n-j}\left\{\frac{\tfrac{3}{4}j^{4/3}(n-j)^{4/3}-1}{n^{4/3}-1}\right\}\tilde{a}_j + \frac{\sum_{k=1}^{j-1}\tilde{a}_k\tilde{a}_{j-k}}{3(n^{4/3}-1)} = \tilde{A}_n\tilde{a}_1{}^n \quad (n \geq 3) \tag{38c}$$

and where the values for $\tilde{A}_n$ are given in Table 2. $\tilde{a}_1$ is a constant determined in conjugation with the downstream flow solution. Once $\tilde{a}_1$ is determined, $\tilde{u}_1\tilde{v}_1$, etc., can be calculated.

Table 2 Coefficients, $\tilde{A}_n$

n	$\tilde{A}_n$	n	$\tilde{A}_n$
1	1.0	6	-6.7813×10^{-5}
2	-1.6449×10^{-1}	7	-2.8170×10^{-5}
3	1.2798×10^{-2}	8	-1.3747×10^{-5}
4	-1.4817×10^{-3}	9	-7.0577×10^{-6}
5	-9.1987×10^{-5}	10	-3.8033×10^{-6}

The flow across the shock is governed by the transonic oblique shock relations

$$\tilde{u}_{1_d} = [2/(\gamma+1)]\tilde{\beta}'^2 - \tilde{u}_{1_u} \tag{39a}$$

$$\tilde{v}_{1_d} - \tilde{v}_{1_u} = [2/(\gamma+1)]\tilde{\beta}'^3 - 2\tilde{u}_{1_u}\tilde{\beta}' \tag{39b}$$

where $\tilde{\beta}'(\eta)$ is the stretched shock angle relative to the vertical and subscripts u and d indicate conditions upstream and downstream of the shock, respectively. Since Eqs. (39) do not define the location of the shock, $(x_s, \tilde{y}_s)$, we also need the equation

$$d\tilde{y}_s/dx_s = -1/\tilde{\beta}'(\eta) \tag{40}$$

The flow downstream of the shock is a simple wave. Because of shock losses, in this downstream region,

$$f_2(\xi) = \tfrac{2}{3}(\gamma+1)^{1/2}\Lambda' < \tfrac{2}{3}(\gamma+1)^{1/2} \tag{41}$$

where Λ' is the shock strength parameter and is given, insofar as the interaction problem is concerned. Equations (33b, 39, and 41) may be combined to form an equation for $\tilde{\beta}$ as a function of $(\tilde{u}_{1_u}, \tilde{v}_{1_u})$, i.e., at a given c^+ characteristic. The resultant equation, with Eq. (40) is used to locate the oblique shock. Equation (41) applies, also, to the expansion fan and trailing characteristics.

Although the undisturbed boundary layer grows indefinitely, the perturbations due to the interaction must return to zero. For example, the flow leaving the expansion fan has a negative V-velocity component; it must be turned to the horizontal again, i.e., so that $\tilde{v}_1(\infty, 0) = 0$.

From Eqs. (33b, 41, and 20c), one can obtain an equation for the pressure at the wall downstream of the shock

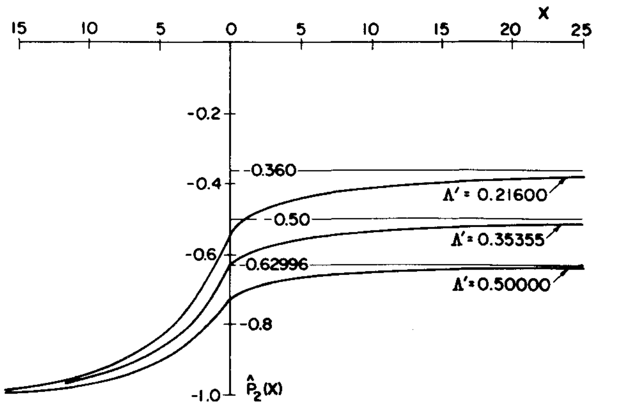

Fig. 2 Perturbation pressure distribution at the wall for various shock strengths.

$$[-\hat{P}_2(x)]^{3/2}+\omega\int_{-\infty}^{x}(x-\zeta)^{-1/3}\frac{d^2\hat{P}_2}{d\zeta^2}\,d\zeta=\Lambda'<1 \quad (x>0) \tag{42}$$

Care must be taken in evaluating the integral in Eq. (42) since $d^2\hat{P}_2/dx^2$ is discontinuous at $x=0$ as shown below.

Equation (42) must be solved numerically, although expansions for $\hat{P}_2(x)$ for large and small x can be derived.[13] Thus

$$\hat{P}_2(x)=-\Lambda'^{2/3}-(2\omega/9\Lambda'^{1/3})[1-\Lambda'^{2/3}]x^{-4/3}+\cdots \quad (x\gg 1) \tag{43a}$$

$$\hat{P}_2(x)=\hat{P}_2(0^-)+\hat{P}_2{}'(0^-)x-[9(3)^{1/2}/8\pi\omega](1-\Lambda')x^{4/3}+\tfrac{1}{2}\hat{P}_2{}''(0^-)x^2+O[(1-\Lambda')x^{8/3}] \quad (0<x\ll 1) \tag{43b}$$

Equation (43b) can be compared with the expansion for small but negative x

$$\hat{P}_2(x)=\hat{P}_2(0^-)+\hat{P}_2{}'(0^-)x+\tfrac{1}{2}\hat{P}_2{}''(0^-)x^2+O(x^3) \tag{44}$$

Thus, it is seen that the second derivative of the pressure is discontinuous at $x=0$, being bounded for $x=0^-$, but infinite for $x=0^+$. The V-velocity component is also discontinuous at $x=0$. This implies the existence of an inner region in the boundary layer such that $\tau_{\text{inner}}\ll\tau$. However, the solution to the oblique shock problem can be completed without studying this inner region. Finally, it should be noted [Eq. (43a)] that the pressure decays algebraically to its final value; this is slow compared to the upstream influence which obeys an exponential law as $x\to-\infty$.

As part of the numerical solution of Eq. (42), $\tilde{a}_1$ (i.e., ω) must be determined. A double iteration scheme, involving both $\tilde{a}_1$ and $\hat{P}_2(x)$ for $x>0$, was employed,[13] with the proper solution being that one which satisfied Eq. (43a). With $\tilde{a}_1$ and $\hat{P}_2(x)$

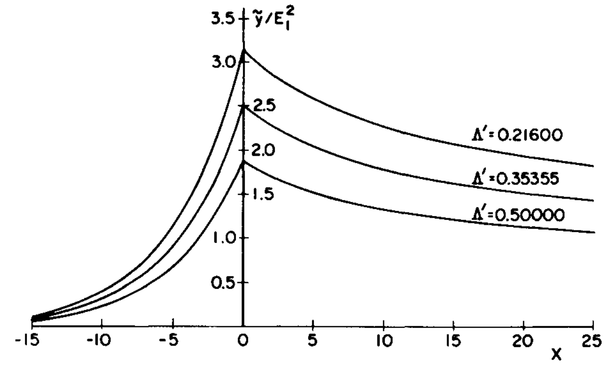

Fig. 4 Shape of wall streamline in transonic flow region for various shock strengths.

known, the remaining properties in the transonic flow region may be found directly.

Calculations were made for three different shock strengths, $\Lambda'=0.21600, 0.35355$, and 0.50000. An attempted calculation for $\Lambda'=0.12500$ resulted in no solution; this will be discussed later. The flow conditions for the incoming shocks corresponding to each Λ' value are shown in Table 3. In each case, $\gamma=1.4$.

Table 3 Conditions downstream of incoming shock wave for various shock strengths[a]

Λ'	$\tilde{\beta}'$	$\tilde{u}_{1_d}$	$\tilde{v}_{1_d}$
0.50000	1.4801	0.82552	−0.25835
0.35355	1.4578	0.77092	−0.33394
0.21600	1.4358	0.71786	−0.40508
0.12500	1.4206	0.68169	−0.45219

[a] For these calculations, $\tilde{u}_{1_u}=1$, $\tilde{v}_{1_u}=0$.

The pressure perturbations are shown in Fig. 2. They are monotonic with most (70%–75%) of the change occurring upstream of the shocks. The V-velocity components at the wall can be found using Eq. (33b). The results are shown in Fig. 3. If the V-velocity component is integrated from $-\infty$ to x, the position of the streamline at $\tilde{y}=0$ (i.e., the wall streamline) is obtained as shown in Fig. 4. In this figure the vertical scale is stretched by a factor of $E_1{}^2$ in order to illustrate the variations which are actually very small. Finally, the numerical solutions may be used to construct the transonic flowfield as shown in Fig. 5 for

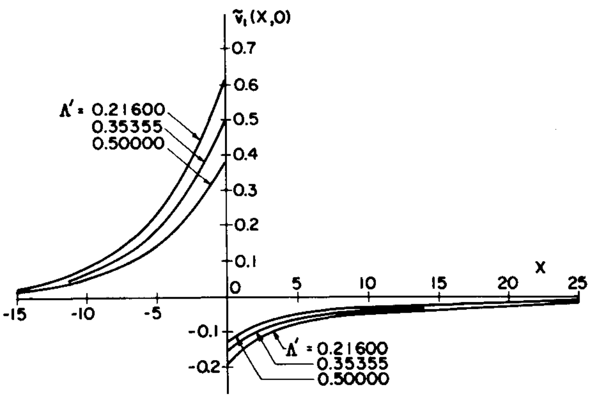

Fig. 3 V-velocity component at the wall, in transonic flow region, for various shock strengths.

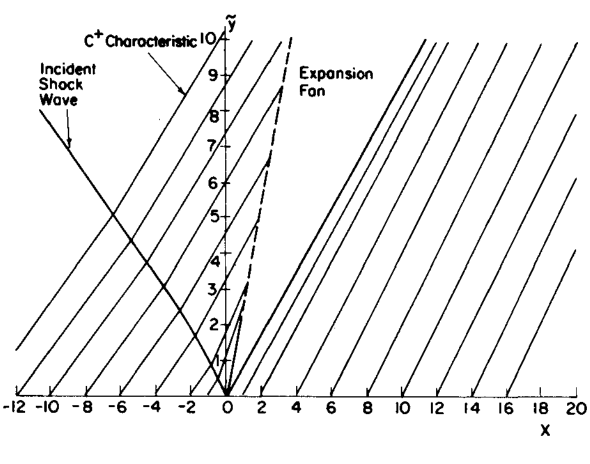

Fig. 5 Transonic flow region for $\Lambda'=0.21600$.

$\Lambda' = 0.216$. The interaction between the compression waves and shock ahead of the intersection point causes the shock to curve and become weaker as the wall is approached. It should be noted that the compression characteristics eventually coalesce to form the reflected shock.

The limit on Λ' occurs when the flow at some point becomes subsonic, since then characteristic theory cannot be applied. The lowest velocity in the flowfield occurs along the leading characteristic of the expansion fan, not far downstream. It can be shown[13] that this velocity becomes sonic when $\Lambda' \approx 0.20$, explaining the aforementioned lack of success in finding a solution for $\Lambda' = 0.125$.

Discussion

The over-all picture of the interaction between a weak shock wave and an unseparated boundary-layer is similar in some respects to that found in the separated case for either transonic or supersonic flow. That is, even though the shock is very weak, the shock thickness is still small compared to the boundary-layer thickness which is in turn small compared to the extent of the interaction region. In addition, the shock impinges on the boundary layer and this is followed immediately by an expansion fan just as in the other cases. Finally, the pressure distribution upstream of the shock is similar. Only the magnitude of the various regions and the pressure distributions downstream of the shock show basic differences.

Although no direct comparison with experiment is possible, comparison of the flow picture in Refs. 1 and 2, for example, with Fig. 5 indicate that the general structure is reproduced here. Thus, in the event that repeated shocks occur in the supersonic region over an airfoil, all the shocks appear to be oblique shocks followed by expansion fans, with the exception of the last shock which is normal.

The interaction between a normal shock and an unseparated boundary layer can be studied using the model presented here. The only changes occur in the transonic region where the flow behind the shock is subsonic. However, the elliptic equations in this subsonic region are very difficult to solve.

In flow over curved surfaces, the effects of wall curvature and nonuniform incoming flow could require changes to the model presented here. However, it can be shown[13] that as long as the wall curvature is at most of order $\varepsilon^{3/2}$, no correction due to wall curvature need be made.

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